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Letter to the Editor

Non-linear vibration analysis of beams by a spline-based differential quadrature method

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1. Introduction

A way to categorize the differential quadrature is based on the selection of trial functions to determine the weighting coefficients. So far, various trial functions have been used to determine the weighting coefficients, among which the Lagrange interpolation functions are widely used for its simplicity and explicitness [1,2]. Although it is well known that Lagrange interpolation functions are limited by the number of interpolation points and severe oscillation may take place if the order is large, the use of the Gausss–Lobatto–Chebyshev points [1] can accelerate the convergence rate of the differential quadrature method in majority cases and accordingly the worsening of the solution is forestalled. But the use of Gausss–Lobatto–Chebyshev points is not a panacea; this might be partially the reason behind the development of harmonic differential quadrature (HDQ) [3] and the spline-based differential quadrature [4] in which much more grid points can be used.

The use of spline functions to determine the weighting coefficients in the differential quadrature method was actually initiated by Kashef and Bellman [5]. In their work, the weighting coefficients were determined using cubic cardinal B-spline functions, but they did not elaborate on the approach and no explicit formulae for the weighting coefficients were given. In this note, a differential quadrature method based on the sextic B-spline functions is developed and explicit formulae to evaluate the weighting coefficients are presented. The non-linear free vibrations of beams with various boundary conditions are studied to validate the new development. The non-linear free vibrations of beams with immovable ends have been tackled using a variety of methods, such as the continuum approach [6–9] and finite element method [10–13]. Feng and Bert also investigated the non-linear vibrations of beams using the conventional DQM [14]. In this note, the same assumptions [14] are made for simply supported beams and the governing equation is solved using the newly developed differential quadrature method. The non-linear vibration analysis of beams with other boundary conditions is also conducted despite the complexity resulting from the change of shape mode of beam with vibration amplitude [7,12,13]. The comparison of computed results with those of other methods

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shows that the differential quadrature method based on the sextic B-spline functions is reliable and effective. The present work also demonstrates the equal usefulness of even-order spline functions.

2. Spline-based differential quadrature

2.1. Cardinal sextic B-spline interpolation

The differential quadrature method based on odd-order B-splines has been elaborated in Ref. [4]. The one major motivation of the development of differential quadrature using the sextic B-spline is that the convergence rate of the differential quadrature method based on the quintic B-spline has been found to be less satisfactory in the case of vibration analysis of Bernoulli–Euler beams. First of all, a set of uniformly spaced knots is selected in a normalized interval [0,1], i.e.,

$$x_0 = 0,\ x_N = 1,\ x_{j+1} - x_j = h,\ \ j = 0,\ 1,\ 2, \ldots, N-1. \tag{1}$$

To construct a global interpolation function over the interval, usually extra knots outside the interval $[x_0, x_N]$ are needed to meet the end condition requirements. A typical spline interpolation over the given interval using the sextic B-spline can be expressed as

$$s_6(x) = \sum_{j=-3}^{N+3} \Phi_j(x) y_j, \quad \Phi_j(x) = \Phi_0(x - jh). \tag{2}$$

In order to meet the required interpolation condition, the interpolation functions $\Phi_j(x)$ should satisfy the cardinal condition at every knot, i.e.,

$$\Phi_j(x_i) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$
$$i,j = -3, -2, -1, 0, 1, \ldots, N-1, N, N+1, N+2, N+3, \tag{3}$$

where $\Phi_j(x)$ are usually given in terms of a combination of translated and scaled spline function φ_6. To acquire the cardinal spline interpolation function $\Phi_j(x)$, the following four auxiliary spline interpolation functions [15] are constructed first:

$$\psi_6(x) = \sum_{j=-3}^{N+3} y_j \varphi_6(x - x_j), \tag{4a}$$

$$\psi_6(x)^{\langle 1/6 \rangle} = \sum_{j=-3}^{N+3} y_j \varphi_6^{\langle 1/6 \rangle}(x - x_j), \tag{4b}$$

$$\psi_6(x)^{\langle 1/3 \rangle} = \sum_{j=-3}^{N+3} y_j \varphi_6^{\langle 1/3 \rangle}(x - x_j), \tag{4c}$$

$$\psi_6(x)^{\langle 1/2 \rangle} = \sum_{j=-3}^{N+3} y_j \varphi_6^{\langle 1/2 \rangle}(x - x_j), \tag{4d}$$

where

$$\varphi_6^{\langle 1/6\rangle}(x) \equiv \varphi_6\left(x+\frac{h}{6}\right)+\varphi_6\left(x-\frac{h}{6}\right), \tag{5a}$$

$$\varphi_6^{\langle 1/3\rangle}(x) \equiv \varphi_6\left(x+\frac{h}{3}\right)+\varphi_6\left(x-\frac{h}{3}\right), \tag{5b}$$

$$\varphi_6^{\langle 1/2\rangle}(x) \equiv \varphi_6\left(x+\frac{h}{2}\right)+\varphi_6\left(x-\frac{h}{2}\right). \tag{5c}$$

φ_6 is the normalized sextic B-spline function [16].

With the local non-zero property of the spline function $\varphi_6(x)$, all the terms but the one containing y_j on the right sides of Eqs. (4) can be eliminated. Thus, the cardinal spline interpolation function is obtained as

$$s_6(x) = \frac{76946}{15}\psi_6(x) - \frac{152469}{40}\psi_6^{\langle 1/6\rangle}(x) + 1485\psi_6^{\langle 1/3\rangle}(x) - \frac{28517}{120}\psi_6^{\langle 1/2\rangle}(x), \quad s_6(x)\in C^5. \tag{6}$$

Hence,

$$\begin{aligned}\Phi_j(x) = & \frac{76946}{15}\varphi_6(x-x_j) - \frac{152469}{40}\varphi_6^{\langle 1/6\rangle}(x-x_j) \\ & + 1485\varphi_6^{\langle 1/3\rangle}(x-x_j) - \frac{28517}{120}\varphi_6^{\langle 1/2\rangle}(x-x_j).\end{aligned} \tag{7}$$

Since the extra knots outside the interval are often cumbersome to handle, non-integral knots within the interval are introduced instead in this note. Namely, $x_{1/6} = h/6$, $x_{1/3} = h/3$, $x_{1/2} = h/2$ and $x_{N-1/2} = (N-1/2)h$, $x_{N-1/3} = (N-1/3)h$, $x_{N-1/6} = (N-1/6)h$ are added in the vicinity of the two ends of the interval. Through some simple mathematic manipulations, the cardinal sextic spline interpolation function can be re-arranged into the following form that is free of extra outside knots:

$$\begin{aligned}& s_6(x) = \sum_{j=0}^{N}\Omega_j(x)y_j, \quad \Omega_j(x_i) = \delta_{ij} = \begin{cases}1, & i=j,\\ 0, & \text{otherwise},\end{cases} \\ & i,j = 0, 1/6, 1/3, 1/2, 1, 2, \ldots, N-2, N-1, N-1/2, N-1/3, N-1/6, N.\end{aligned} \tag{8}$$

2.2. Weighting coefficients

All weighting coefficients are given in explicit forms:

$$\begin{aligned}& C_{ij}^{(n)} = \Omega_j^{(n)}(x_i), \quad n = 1, \ldots, 6 \\ & i,j = 0, 1/6, 1/3, 1/2, 1, 2, \ldots, N-2, N-1, N-1/2, N-1/3, N-1/6, N.\end{aligned} \tag{9}$$

The localized non-zero nature of splines results in banded weighting coefficient matrices for derivatives. As reported in Ref. [4], this nature enables the differential quadrature to tackle problems with local discontinuity.

3. Non-linear vibrations of beams

A Bernoulli–Euler beam oscillating with large amplitude on immovable ends is considered here. The governing equation for non-linear vibrations of beams can be described as [13]

$$EI\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0, \tag{10}$$

where w and m represent the deflection and the mass density per unit length. Assuming that the ends are axially immovable, i.e., $u(0,t) = u(L,t) = 0$, it is evident that the axial force N is independent of x and thus depends only on time [13],

$$N(x,t) = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = N(t) = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x. \tag{11}$$

For a simply supported beam, it is reasonable to assume that [14]

$$w(x,t) = av(x)\cos\omega t. \tag{12}$$

The governing equation for simply supported beams can be developed using the Ritz–Galerkin technique [18], i.e.,

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - \frac{3}{4}\left[\frac{EAa^2}{2L}\int_0^L \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \mathrm{d}x\right]\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \omega^2 mv = 0. \tag{13}$$

Its dimensionless form is

$$\frac{\mathrm{d}^4 v}{\mathrm{d}\xi^4} - \frac{3}{4}\left[\frac{a^2}{2r^2}\int_0^1 \left(\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)^2 \mathrm{d}x\right]\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - (\omega^*)^2 mv = 0, \tag{14}$$

where

$$\xi = \frac{x}{L}, \quad (\omega^*)^2 = \omega^2\frac{mL^4}{EI}, \quad r^2 = \frac{I}{A}. \tag{15}$$

For a simply supported beam, the boundary conditions can be written as

$$v(0) = \frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2}(0) = v(1) = \frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2}(1) = 0. \tag{16}$$

As noted by some researchers [7,12,13], within the framework of the moderately large bending theory, the non-linear vibration of simply supported beams would admit a variable-separable solution, but the beams with clamped ends would not. For beams with clamped end, it is usually assumed [13] that a point of maximum amplitude exists during the vibration and that this is also the point of reversal of motion of every point of the beam. The properties of the eigenvector and time function at the point are chosen as in Ref. [13]. Assume that at the point where maximum amplitude is reached, the configuration of the beam is represented by $\bar{w}$ and there exists [13]

$$\frac{\partial^2 \bar{w}}{\partial t^2} = -\omega^2\bar{w}, \quad \frac{\partial \bar{w}}{\partial t} = 0. \tag{17}$$

Substituting the above expression into governing equation (10) results in the differential equation for non-linear vibrations of beams with clamped ends

$$\frac{\mathrm{d}^4\bar{w}}{\mathrm{d}\xi^4} - \left[\frac{a^2}{2r^2}\int_0^1\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}\right)^2\mathrm{d}x\right]\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2} - (\omega^*)^2 m\bar{w} = 0, \tag{18}$$

where ξ, $(\omega^*)^2$ and r^2 are given in Eq. (15). Applying the differential quadrature rule to Eqs. (14) and (18) and invoking the boundary conditions at the two ends yields a set of algebraic equations.

The resultant non-linear eigenvalue problem is solved through an iterative scheme [14]. It is found that in all computed cases not more than three iterations are needed to acquire convergent eigenvalues and eigenvectors.

4. Results and discussion

The convergence of the newly developed differential quadrature method is examined first. The first five linear natural frequencies of a simply supported beam are obtained and displayed in Table 1. For the fundamental frequency ω_L, excellent agreement with the exact value is reached when N is increased to 15. The five frequencies are all in excellent agreement with their exact values when N is increased to 50. Large numbers of knots, $N = 200$ or 300, say, have also been used to demonstrate the stability of the method. It is seen that the differential quadrature based on the spline functions is very stable in comparison with the conventional DQM whose grid number is usually restricted to below 30.

Tables 2–4 show the variation of the non-linear frequency ratio ω/ω_L with amplitude of vibration a/r, for three kinds of boundary conditions. In Table 2, the results of non-linear frequency ratio ω/ω_L for a simply supported beam are set out and compared with the exact solution [18], conventional DQ results [14] and FEM results [10]. The present results are in

Table 1
Convergence study: the first five linear natural frequencies of simply supported beams

N	Mode sequence				
	1	2	3	4	5
6	9.8693	39.2265	84.3757	133.4133	171.3206
8	9.8689	39.3957	87.4887	149.4603	215.7533
10	9.8691	39.4395	88.2796	154.4341	233.0186
15	9.8695	39.4683	88.7045	157.1905	243.8744
20	9.8695	39.4747	88.7833	157.6662	245.7799
30	9.8696	39.4776	88.8167	157.8589	246.5308
40	9.8696	39.4781	88.8231	157.8952	246.6696
50	9.8696	39.4783	88.8250	157.9058	246.7101
100	9.8696	39.4784	88.8263	157.9131	246.7381
200	9.8696	39.4784	88.8264	157.9136	246.7400
300	9.8696	39.4784	88.8264	157.9137	246.7401
Exact [17]	9.8696	39.4784	88.8264	157.9137	246.7401

Table 2
Ratio of the non-linear frequency to the linear frequency (ω/ω_L) for simply supported beams

a/r	Exact [18]	Present	DQ [14]	FEM [10]
0.1	1.0009	1.0009	1.0010	1.0009
0.2	1.0037	1.0037	1.0043	1.0037
0.4	1.0149	1.0149	1.0170	1.0148
0.6	1.0332	1.0332	1.0384	1.0339
0.8	1.0583	1.0583	1.0673	1.0578
1.0	1.0897	1.0897	1.1030	1.0889
1.5	1.1924	1.1924	1.2045	1.1902
2.0	1.3229	1.3229	1.3170	1.3022

Table 3
Ratio of the non-linear frequency to the linear frequency (ω/ω_L) for clamped beam

a/r	GFEM [13]	Present	DQ [14]	FEM [10]	ASM [7]
0.1	1.0003	1.0003	1.0003	1.0003	1.0003
0.2	1.0012	1.0012	1.0011	1.0012	1.0012
0.4	1.0048	1.0048	1.0044	1.0048	1.0048
0.6	1.0107	1.0108	1.0100	1.0107	1.0107
0.8	1.0190	1.0190	1.0178	1.0190	1.0190
1.0	1.0295	1.0296	1.0278	1.0295	1.0296
1.5	1.0650	1.0652	1.0628	1.0650	1.0653
2.0	1.1127	1.1129	1.1119	1.1127	1.1135

Table 4
Ratio of the non-linear frequency to the linear frequency (ω/ω_L) for beams with one end simply supported the other end clamped

a/r	GFEM [13]	Present	FEM [10]	ASM [7]
0.1	1.0006	1.0006	1.0006	1.0006
0.2	1.0026	1.0024	1.0026	1.0026
0.4	1.0106	1.0097	1.0106	1.0106
0.6	1.0237	1.0218	1.0237	1.0238
0.8	1.0416	1.0383	1.0416	1.0418
1.0	1.0641	1.0592	1.0641	1.0647
1.5	1.1378	1.1284	1.1378	1.1404
2.0	1.2318	1.2179	1.2319	1.2385

excellent agreement with the exact solutions. Tables 3 and 4 present results for beams with two ends clamped and beams with one end simply supported and the other end clamped. The present results and those of other methods are also in very good agreement, further verifying the effectiveness of the present method.

5. Conclusions

Based on sextic cardinal spline functions, a spline-based differential quadrature method is developed and successfully applied to the solution of higher order non-linear differential equations. In the analysis of the non-linear vibrations of beams, the computed results are found to be in excellent agreement with those of the exact solutions and FEM results, indicating that the present method is effective. The spline-based differential quadrature has been shown to be very stable; it therefore can be counted on to deal with other problems in addition to the non-linear vibration of beams. The present work also demonstrates the equal usefulness of even-order spline functions in practice.

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